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An Extremal Problem for Sets with Applications to Graph Theory

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Let $X_1,..., X_n$ be n disjoint sets. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant h$ let A_{ij} and B_{ij} be subsets of X_i that satisfy $|A_{ij}| \leqslant r_i$ and $|B_{ij}| \leqslant s_i$ for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant h$, $(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ij}) = \varnothing$ for $1 \leqslant j \leqslant h$, $(\bigcup_i A_{ij}) \cap (\bigcup_i B_{il}) \neq \varnothing$ for $1 \leqslant j < l \leqslant h$. We prove that $h \leqslant \prod_{i=1}^n \binom{r_i+s_i}{r_i}$. This result is best possible and has some interesting consequences. Its proof uses multilinear techniques (exterior algebra). © 1985 Academic Press, Inc.

1. INTRODUCTION

Our main result in this paper is the following.

THEOREM 1.1. *Let $X_1,..., X_n$ be n disjoint sets and let $r_1,..., r_n$ and $s_1,..., s_n$ be positive integers. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant h$ let A_{ij} and B_{ij} be subsets of X_i that satisfy*

$$|A_{ij}| \leqslant r_i \quad \text{and} \quad |B_{ij}| \leqslant s_i \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant h. \tag{1.1}$$

$$\left(\bigcup_i A_{ij}\right) \cap \left(\bigcup_i B_{ij}\right) = \varnothing \quad \text{for} \quad 1 \leqslant j \leqslant h. \tag{1.2}$$

$$\left(\bigcup_i A_{ij}\right) \cap \left(\bigcup_i B_{il}\right) \neq \varnothing \quad \text{for} \quad 1 \leqslant j < l \leqslant h. \tag{1.3}$$

Then

$$h \leqslant \prod_{i=1}^n \binom{r_i+s_i}{r_i}. \tag{1.4}$$

This result is easily seen to be best possible and it clearly implies the following weaker assertion.

COROLLARY 1.2. *For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant h$ let X_i, s_i, r_i, A_{ij}, and B_{ij} satisfy the hypotheses of Theorem 1.1 and assume, in addition, that*

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$(\bigcup_i A_{ij}) \cap (\bigcup_i B_{il}) \neq \varnothing$ for $1 \leqslant l < j \leqslant h$ (i.e., $(\bigcup_i A_{ij}) \cap (\bigcup_i B_{il}) = \varnothing$ iff $j = l$). Then (1.4) *holds.*

Special cases of Theorem 1.1 and Corollary 1.2 were proved by several authors. Corollary 1.2 with $n=1$ was proved by Bollobás [2], and rediscovered by Jaeger and Payan [8] and by Katona [12]. Theorem 1.1 with $n=1$ was proved by Frankl [7] by modifying an argument of Lovász [13] and was also proved in an equivalent form by Kalai [10].

As we shall see in Section 3, a special case of Corollary 1.2 with $n=2$ was conjectured by Erdös, Hajnal, and Moon [6] and proved by Bollobás [3] and by Wessel [15].

The proof of Theorem 1.1 uses multilinear techniques (exterior algebra). Similar methods were used by Lovász in [13] and by Kalai in [10, 11]. It is worth noting that we can prove Corollary 1.2 in a purely combinatorial manner, but we do not know any such proof to (the stronger) Theorem 1.1 (even for $n=1$).

Theorem 1.1 has many interesting applications. After proving it in the next section we use it in Section 3 to obtain some extensions of results of Bollobás, Erdös, Hajnal, Moon, Kalai, and Wessel on saturated graphs and hypergraphs.

In a forthcoming paper [1] of Kalai and the present author we show how the case $n=1$ in Theorem 1.1 supplies a short proof of the well-known upper bound theorem for polytopes and of some related results in convexity.

2. The Proof of the Main Result

We begin with a brief summary of the algebraic background needed. More details about exterior algebra can be found, e.g., in [14].

Let $V = R^m$ be the m-dimensional real space with the standard basis $e_1, ..., e_m$. Put $M = \{1, 2, ..., m\}$. The exterior algebra $\bigwedge V$ is a 2^m-dimensional real space, (in which V is embedded), equipped with a multilinear associative multiplication $\wedge$. For $S = \{i_1, i_2, ..., i_s\} \subset M$, with $i_1 < i_2 < \cdots < i_s$ put $e_S = e_{i_1} \wedge \cdots \wedge e_{i_s}$. The set $\{e_S : S \subset M\}$ forms a basis of $\bigwedge V$. For $0 \leqslant k \leqslant m$, $\bigwedge^k V$ is the $\binom{m}{k}$-dimensional subspace of $\bigwedge V$ spanned by $\{e_S : S \subset M, |S| = k\}$.

Our proof uses the following property of the $\wedge$ product. Suppose $r + s = m$ and let $v_1, ..., v_r, u_1, ..., u_s \in V$. Define $v = v_1 \wedge v_2 \wedge \cdots \wedge v_r \in \bigwedge^r V$ and $u = u_1 \wedge \cdots \wedge u_s \in \bigwedge^s V$. Then $u \wedge v \neq 0$ if and only if $v_1, ..., v_r, u_1, ..., u_s$ are independent in V. In particular, if $\{v_1, ..., v_r\} \cap \{u_1, ..., u_s\} \neq \varnothing$ then $u \wedge v = 0$.

Proof of Theorem 1.1. Clearly we may assume that $|A_{ij}| = r_i$ and $|B_{ij}| = s_i$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant h$. For $1 \leqslant i \leqslant n$ let $V_i = R^{r_i + s_i}$ be an $(r_i + s_i)$-

dimensional real space. Let $\{z_i^d : d \in (\bigcup_j A_{ij}) \cup (\bigcup_j B_{ij})\}$ be vectors in general position in V_i (i.e., every $r_i + s_i$ of these vectors are independent in V_i).

Consider the following two $\prod_{i=1}^n \binom{r_i+s_i}{r_i}$-dimensional real subspaces of the exterior algebra $\bigwedge (V_1 \oplus \cdots \oplus V_n)$ in which each V_i is naturally imbedded:

$$V = \left(\bigwedge^{r_1} V_1\right) \wedge \left(\bigwedge^{r_2} V_2\right) \wedge \cdots \wedge \left(\bigwedge^{r_n} V_n\right) = \bigwedge_{i=1}^{n} \left(\bigwedge^{r_i} V_i\right)$$

and

$$\bar{V} = \left(\bigwedge^{s_1} V_1\right) \wedge \left(\bigwedge^{s_2} V_2\right) \wedge \cdots \wedge \left(\bigwedge^{s_n} V_n\right) = \bigwedge_{i=1}^{n} \left(\bigwedge^{s_i} V_i\right).$$

For $1 \leqslant j \leqslant h$ define

$$y_j = \bigwedge_{i=1}^{n} \left(\bigwedge_{t \in A_{ij}} z_i^t \right) \in V$$

and

$$\bar{y}_j = \bigwedge_{i=1}^{n} \left(\bigwedge_{q \in B_{ij}} z_i^q \right) \in \bar{V}.$$

Note that the properties of the $\wedge$-product, (1.2) and the general position of the z_i^d's imply that

$$y_j \wedge \bar{y}_j \neq 0 \qquad \text{for} \quad 1 \leqslant j \leqslant h. \tag{2.1}$$

Similarly (1.3) implies that

$$y_j \wedge \bar{y}_l = 0 \qquad \text{for} \quad 1 \leqslant j < l \leqslant h. \tag{2.2}$$

To complete the proof we show that the set $\{y_j : 1 \leqslant j \leqslant h\}$ is linearly independent in V and thus $h \leqslant \dim V = \prod_{i=1}^n \binom{r_i+s_i}{r_i}$. Indeed, suppose this is false and let

$$\sum_{j \in J} c_j y_j = 0 \tag{2.3}$$

be a linear dependence, with $c_j \neq 0$ for $j \in J$. Put $l = \max\{j : j \in J\}$. Combining (2.2) and (2.3) we obtain $0 = (\sum_{j \in J} c_j y_j) \wedge \bar{y}_l = \sum_{j \in J} c_j(y_j \wedge \bar{y}_l) = c_l(y_l \wedge \bar{y}_l)$, which together with (2.1) supplies the contradiction $c_l = 0$. This completes the proof. ∎

3. Saturated Graphs and Hypergraphs

In this section we use Theorem 1.1 to obtain some results about saturated graphs and hypergraphs. Some other results in this direction can be found in [11]. We begin with some notation and definitions.

Let $e(G)$ denote the number of edges of a graph G and let $v(G)$ denote the number of its vertices. For a graph H let $N(G, H)$ denote the number of subgraphs of G isomorphic to H. K_l is the complete graph on l vertices, $K_{l,m}$ is the complete bipartite graph with l vertices in one class and m in the other, and $G_2(l, m)$ denotes a bipartite graph with l vertices in one class and m in the other.

Following Bollobás [5, pp. 308, 325, 362] we say that G is *strongly H-saturated* if $N(G, H) < N(G^+, H)$ whenever G^+ is obtained from G by the addition of an edge. We say that G is *weakly H-saturated* if there is a sequence of graphs $G = G_0 \subset G_1 \subset \cdots \subset G_t = K_{v(G)}$ such that $e(G_i) = e(G_{i-1}) + 1$ and $N(G_i, H) > N(G_{i-1}, H)$ for $1 \leqslant i \leqslant t$. Clearly, every strongly H-saturated graph is also weakly H-saturated.

Similarly, if $H = (V, E)$ is bipartite with bipartition $V = V_1 \cup V_2$ then $G = G_2(l, m)$ is *strongly H-bisaturated* if the number of H's in G (with V_1 in the first class of vertices of G and V_2 in the second) increases whenever we add an edge joining vertices belonging to different classes. G is *weakly H-bisaturated* if there is a sequence of graphs $G = G_0 \subset G_1 \subset \cdots \subset G_t = K_{l,m}$, such that G_i is obtained from G_{i-1} by adding an edge joining vertices belonging to different classes and there exists a copy of H in G_i (with V_1 in the first class of vertices and V_2 in the second) which is not in G_{i-1} $(1 \leqslant i \leqslant t)$.

Let $f_s(l, H)$ $(f_w(l, H))$ denote the minimal possible number of edges in a strongly H-saturated (weakly H-saturated) graph on l vertices. Similarly, for a bipartite graph H let $g_s(l, m, H)$ $(g_w(l, m, H))$ denote the minimal possible number of edges in a strongly H-bisaturated (weakly H-bisaturated) graph $G = G_2(l, m)$.

The main result of Erdös, Hajnal, and Moon in [6] is that for $3 \leqslant r \leqslant l$

$$f_s(l, K_r) = \binom{l}{2} - \binom{l-r+2}{2} \qquad \left(= (r-2) \cdot l - \binom{r-1}{2} \right). \tag{3.1}$$

This was generalized by Bollobás to hypergraphs in [2]. The authors of [6] conjectured that for $2 \leqslant r \leqslant l$, $2 \leqslant t \leqslant m$,

$$g_s(l, m, K_{r,t}) = l \cdot m - (l-r+1)(m-t+1). \tag{3.2}$$

This was proved by Bollobás [3] and Wessel [15]. In [4] (see also [5, p. 362]) Bollobás conjectured that

$$f_w(l, K_r) = \binom{l}{2} - \binom{l-r+2}{2} \qquad (= f_s(l, K_r)). \tag{3.3}$$

This conjecture was proved by Kalai in [11] using exterior algebra and in [9] by showing that every weakly K_r-saturated graph is $(r-2)$-rigid.

Here we show that all these results are easy consequences of Theorem 1.1. We also prove the following theorem that extends (3.2) analogously to the way (3.3) extends (3.1).

THEOREM 3.1. *For* $2 \leqslant r \leqslant l$ *and* $2 \leqslant t \leqslant m$,

$$g_w(l, m, K_{r,t}) = l \cdot m - (l-r+1)(m-t+1) \qquad (= g_s(l, m, K_{r,t})). \quad (3.4)$$

We further show how all these results can be generalized to hypergraphs. We close the paper with a proof of the following theorem.

THEOREM 3.2. (i) *For every fixed graph* H *the limit* $\lim_{l \to \infty} f_w(l, H)/l$ *exists.*

(ii) *For every fixed bipartite graph* H *and every fixed* m, *the limit* $\lim_{l \to \infty} g_w(l, m, H)/l$ *exists.*

It is worth noting that all the above-mentioned results about strongly saturated graphs are consequences of Corollary 1.2 and thus can be proved without any algebraic tools. The corresponding results for weakly saturated graphs ((3.3) and (3.4)) follow from Theorem 1.1 and we do not know how to prove any of them in a pure combinatorial method.

Proof of (3.1) *and of* (3.3). Since every strongly K_r-saturated graph is also weakly K_r-saturated and since the graph $G = (V, E)$ with $V = \{1, 2, ..., l\}$ and $E = \{ij: \{i, j\} \cap \{1, 2, ..., r-2\} \neq \varnothing\}$ is strongly K_r-saturated, we conclude that

$$f_w(l, K_r) \leqslant f_s(l, K_r) \leqslant \binom{l}{2} - \binom{l-r+2}{2}.$$

In order to complete the proof we must show that

$$f_w(l, K_r) \geqslant \binom{l}{2} - \binom{l-r+2}{2}.$$

Let G be a weakly K_r-saturated graph on the vertices $V = \{1, 2, ..., l\}$. Suppose $G = G_0 \subset G_1 \subset \cdots \subset G_t = K_l$ is a sequence of graphs, where G_j is obtained from G_{j-1} by adding the edge e_j and let K_r^j be the set of vertices of a copy of K_r included in G_j but not in G_{j-1} $(1 \leqslant j \leqslant t)$. One can easily check that $n = 1$, $X_1 = V$, $r_1 = l - r$, $s_1 = 2$, $h = t$, and $A_{1j} = V \backslash K_r^j$, $B_{1j} = e_j$ $(1 \leqslant j \leqslant h)$ satisfy the hyotheses of Theorem 1.1. Therefore $t \leqslant \binom{l-r+2}{2}$ and the number of edges of G is $\geqslant \binom{l}{2} - \binom{l-r+2}{2}$, as needed. ∎

Proof of (3.2) *and of Theorem* 3.1. Since every strongly $K_{r,t}$-bisaturated graph is also weakly $K_{r,t}$-bisaturated and since the graph $G = G_2(l, m)$ with classes of vertices $\{y_1, ..., y_l\}$ and $\{z_1, ..., z_m\}$ in which $y_i z_j$ is an edge iff $i \leqslant r-1$ or $j \leqslant t-1$ is strongly $K_{r,t}$-bisaturated, we conclude that

$$g_w(l, m, K_{r,t}) \leqslant g_s(l, m, K_{r,t}) \leqslant l \cdot m - (l-r+1) \cdot (m-t+1).$$

In order to complete the proof we must show that

$$g_w(l, m, K_{r,t}) \geqslant l \cdot m - (l-r+1) \cdot (m-t+1).$$

Let $G = G_2(l, m)$ be a weakly $K_{r,t}$-bisaturated graph on the classes of vertices $V_1 = \{y_1, ..., y_l\}$ and $V_2 = \{z_1, ..., z_m\}$. Suppose $G = G_0 \subset G_1 \subset \cdots \subset G_h = K_{l,m}$ is a sequence of graphs, where G_j is obtained from G_{j-1} by adding the edge e_j that joins a vertex of V_1 to a vertex of V_2 and let $K^j_{r,t}$ be the set of vertices of a copy of $K_{r,t}$ (with r vertices in V_1 and t in V_2) included in G_j but not in G_{j-1} $(1 \leqslant j \leqslant h)$. One can easily check that $n = 2$, $X_1 = V_1$, $X_2 = V_2$, $r_1 = l - r$, $r_2 = m - t$, $s_1 = s_2 = 1$, and $A_{ij} = V_i \backslash K^j_{r,t}$, $B_{ij} = V_i \cap e_j$ for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant h$, satisfy the hypotheses of Theorem 1.1. Therefore $h \leqslant \binom{l-r+1}{1} \cdot \binom{m-t+1}{1} = (l-r+1) \cdot (m-t+1)$ and the number of edges of G is at least $l \cdot m - (l-r+1) \cdot (m-t+1)$. This completes the proof. ∎

Remark 3.3. Although $f_w(l, K_r) = f_s(l, K_r)$, the extremal examples are not the same and there are many weakly K_r-saturated graphs with $f_w(l, K_r)$ edges that are not strongly K_r-saturated. The same holds also for the bisaturated case. It is also worth noting that neither $f_w(l, H) = f_s(l, H)$ nor $g_w(l, m, H) = g_s(l, m, H)$ holds in general.

We now consider the corresponding problems for hypergraphs. Since the proofs are completely analogous to those given above we just state the results and leave the detailed proofs to the reader.

Let $K^{(k)}_r$ denote the k-uniform complete hypergraph on r vertices and let $K^{(k)}_{r_1, ..., r_k}$ denote the k-uniform complete k-partite hypergraph with r_i vertices in the ith class $(1 \leqslant i \leqslant k)$. Let $G_k(m_1, m_2, ..., m_k)$ denote a k-uniform k-partite hypergraph with m_i vertices in the ith class $(1 \leqslant i \leqslant k)$.

A k-uniform hypergraph G is *strongly* $K^{(k)}_r$ *saturated* if the number of copies of $K^{(k)}_r$ in G increases whenever we add an edge to G. G is *weakly* $K^{(k)}_r$ *saturated* if there is a sequence of graphs $G = G_0 \subset G_1 \subset \cdots \subset G_t = K^{(k)}_l$, where l is the number of vertices of G, G_i is obtained from G_{i-1} by the addition of an edge, and G_i contains at least one more copy of $K^{(k)}_r$ than G_{i-1} $(1 \leqslant i \leqslant t)$. Let $F_s(l, K^{(k)}_r)$ $(F_w(l, K^{(k)}_r))$ denote the minimal possible number of edges in a strongly $K^{(k)}_r$ saturated (weakly $K^{(k)}_r$ saturated) k-uniform hypergraph on l vertices.

Similarly $G = G_k(m_1, \ldots, m_k)$ is *strongly-$K^{(k)}_{r_1,\ldots,r_k}$ k-saturated* if the number of copies of $K = K^{(k)}_{r_1,\ldots,r_k}$ in G (with r_i vertices in the ith class) increases whenever we add to G an edge containing one vertex from each class of vertices of G. G is *weakly K-k-saturated* if there is a sequence of graphs $G = G_0 \subset G_1 \subset \cdots \subset G_t = K^{(k)}_{m_1,\ldots,m_k}$ such that G_i is obtained from G_{i-1} by the addition of an edge containing one vertex from each vertex class of G_{i-1}, and there exists a copy of K in G_i (with r_i vertices in the ith class) which is not included in G_{i-1} $(1 \leqslant i \leqslant t)$. Let $G_s(m_1, \ldots, m_k, K^{(k)}_{r_1,\ldots,r_k})$ $(G_w(m_1, \ldots, m_k, K^{(k)}_{r_1,\ldots,r_k}))$ denote the minimal possible number of edges in a strongly K-k-saturated (weakly-K-k-saturated) hypergraph $G = G_k(m_1, \ldots, m_k)$.

The next theorem generalizes (3.1), (3.2), (3.3), and (3.4). $F_s(l, K^{(k)}_r)$ was determined by Bollobás in [2].

THEOREM 3.4. (i) *For* $l \geqslant r \geqslant k$,

$$F_s(l, K^{(k)}_r) = F_w(l, K^{(k)}_r) = \binom{l}{k} - \binom{l-r+k}{k}.$$

(ii) *For* $m_i \geqslant r_i \geqslant k$, $i = 1, \ldots, k$,

$$G_s(m_1, \ldots, m_k, K^{(k)}_{r_1,\ldots,r_k}) = G_w(m_1, \ldots, m_k, K^{(k)}_{r_1,\ldots,r_k})$$
$$= \prod_{i=1}^{k} m_i - \prod_{i=1}^{k} (m_i - r_i + 1).$$

Returning to graphs we close this paper with a proof of Theorem 3.2.

Proof of Theorem 3.2. (i) Suppose $H = (V, E)$ and let $h = |V(H)| - 2$. We first show that for every $l, m \geqslant h$

$$f_w(l+m, H) \leqslant f_w(l, H) + f_w(m, H) + h^2. \tag{3.5}$$

Indeed let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two weakly H-saturated graphs with $|V_1| = l$, $|V_2| = m$, $|E_1| = f_w(l, H)$, $|E_2| = f_w(m, H)$. Let $G = (V, E)$ be the graph obtained from the disjoint union of G_1 and G_2 by joining each of the vertices of an h-subset $U_1 \subset V_1$ to each of the vertices of an h-subset $U_2 \subset V_2$. Clearly $|V| = l + m$ and $|E| = f_w(l, H) + f_w(m, H) + h^2$. To prove (3.5) we show that G is weakly H-saturated. By the choice of G_1 and G_2 one can add edges successively to G to obtain a graph G' such that $G'[V_1]$ and $G'[V_2]$ are complete and each addition increases the number of copies of H. Now we can add, in an arbitrary order, all the edges that join a vertex of U_1 to a vertex of V_2; each such addition adds a new copy of K_{h+2} and thus, clearly, also a new copy of H. We now add all missing edges in an arbitrary order. Since now U_1 is adjacent to all vertices each such addition adds a new copy of K_{h+2} and thus also of H. This completes the proof of (3.5).

Define $g(n)=f_w(n, H)+h^2$. By (3.5) for $l, m \geqslant h$,

$$g(l+m) \leqslant g(l)+g(m),$$

i.e., g is subadditive. Therefore, as is well known, the limit $\lim_{l \to \infty} g(l)/l$ exists and thus the limit $\lim_{l \to \infty} f_w(l, H)/l$ exists, as needed.

It is worth noting that similar construction to the one we used shows that for $l, m \geqslant h$, $f_w(l+m-h, H) \leqslant f_w(l, H)+f_w(m, H)$. Using this one can show that $g(n)=f_w(n+h, H)$ is subadditive and obtain a similar proof of (i).

(ii) Here it is very easy to show that the function $g(n)=g_w(n, m, H)$ is subadditive. The result follows as before. ▮

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